

Novel method of fractal approximation

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Abstract

We introduce new method of optimization for finding free parameters of affine iterated function systems (IFS), which are used for fractal approximation. We provide the comparison of effectiveness of fractal and quadratic types of approximation, which are based on a similar optimization scheme, on the various types of data: polynomial function, DNA primary sequence, price graph and graph of random walking.

1 Introduction

It is well known that approximation is a crucial method for making complicated data easier to describe and operate. In many cases we have to deal with irregular forms, which can't be approximate with desired precision. Fractal approximation become a suitable tool for that purpose. Ideas for interpolation and approximation with the help of fractals appeared in works of M. Barnsley [2] and was developed by P. Massopust [6] and C. Bandt and A. Kravchenko [1].

Today we can apply fractals to approximate such interesting and interdisciplinary data as graphs of DNA primary sequences of different species and interbeat heart intervals [7], price waves and many others.

Section 2 of this work is devoted to the construction of fractal interpolation functions. Necessary condition on free parameters d_i of affine iterated function systems is shown. One graphical example is given.

In section 3 we give the common scheme of approximation of general function $g \in L^2[a, b]$ and obtain the equation for direct calculation of free parameters d_i .

In section 4 we illustrate the results on concrete examples.

2 Fractal Interpolation Functions

There are two methods for constructing fractal interpolation functions. In 1986 M. Barnsley [2] defined such functions, as attractors of some specific iterated function systems. In this work we use common approach, which was developed by P. Massopust [6].

Let $[a, b] \subset \mathbb{R}$ be a nonempty interval, $1 < N \in \mathbb{N}$ and $\{(x_i, y_i) \in [a, b] \times \mathbb{R} \mid a = x_0 < x_1 < \dots < x_{N-1} < x_N = b\}$ — are points of interpolation. For all

$i = \overline{1, N}$ consider affine transformations of the plane

$$A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad A_i \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

We require following two conditions hold true for all i :

$$A_i(x_0, y_0) = (x_{i-1}, y_{i-1}), \quad A_i(x_N, y_N) = (x_i, y_i).$$

In this case

$$\begin{aligned} a_i &= \frac{x_i - x_{i-1}}{b - a}, & c_i &= \frac{y_i - y_{i-1} - d_i(y_N - y_0)}{b - a}, \\ e_i &= \frac{bx_{i-1} - ax_i}{b - a}, & f_i &= \frac{by_{i-1} - ay_i - d_i(by_0 - ay_N)}{b - a}, \end{aligned} \quad (1)$$

there $\{d_i\}_{i=1}^N$ act like family of parameters. Notice, that for all i operator A_i takes the line segment between (x_0, y_0) and (x_N, y_N) to the line segment passes through points of interpolation (x_{i-1}, y_{i-1}) and (x_i, y_i) .

Let \mathcal{K} be a space of nonempty compact subsets \mathbb{R}^2 with Hausdorff metric. Define the Hutchinson operator [5]

$$\Phi : \mathcal{K} \rightarrow \mathcal{K}, \quad \Phi(E) = \bigcup_{i=1}^N A_i(E).$$

It is easily seen [2], that the Hutchinson operator Φ take a graph of any continuous function on a segment $[a, b]$ to a graph of a continuous function on the same segment. Thus, Φ can be treated as operator on the space of continuous functions $C[a, b]$.

For all $i = \overline{1, N}$ denote

$$\begin{aligned} u_i : [a, b] &\rightarrow [x_{i-1}, x_i], & u_i(x) &:= ax + e_i, \\ p_i : [a, b] &\rightarrow \mathbb{R}, & p_i(x) &:= cx + f_i. \end{aligned} \quad (2)$$

Massopust [6] has shown, that Φ acts on $C[a, b]$ according to the rule

$$(\Phi g)(x) = \sum_{i=1}^N ((p_i \circ u_i^{-1})(x) + d_i(g \circ u_i^{-1})(x)) \chi_{[x_{i-1}, x_i]}(x). \quad (3)$$

Moreover, if $|d_i| < 1$ for all $i = \overline{1, N}$, then operator Φ is contractive on the Banach space $(C[a, b], \|\cdot\|_\infty)$ with contractive constant $d \leq \max\{|d_i| \mid i = \overline{1, N}\}$. By the fixed-point theorem there exists unique function $g^* \in C[a, b]$, such that $\Phi g^* = g^*$ and for all $g \in C[a, b]$ we have

$$\lim_{n \rightarrow \infty} \|\Phi^n(g) - g^*\|_\infty = 0.$$

We will call g^* fractal interpolation function. It is clear, that if $g \in C[a, b]$, $g(x_0) = y_0$ and $g(x_N) = y_N$, then $\Phi(g)$ passes through points of interpolation. In this case we will call $\Phi^n(g)$ pre-fractal interpolation functions of order n .

Example 1 Picture shows fractal interpolation function, which was constructed on points of interpolation $(0, 0)$, $(0.5, 0.5)$ $(1, 0)$ with parameters $d_1 = d_2 = 0.5$.

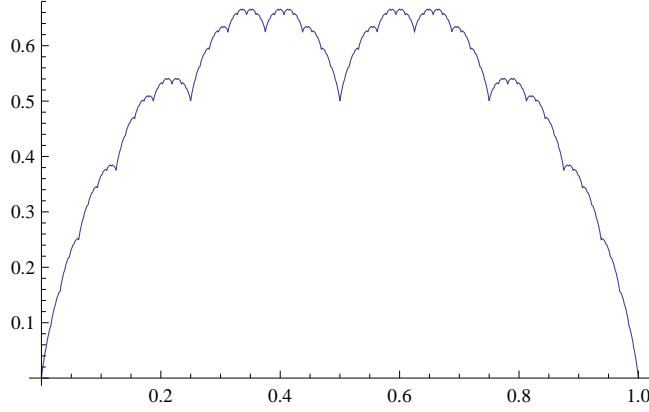


Figure 1: Fractal interpolation function.

3 Approximation

From now on we assume, that $|d_i| < 1$ for all $i = \overline{1, N}$. We try to approximate function $g \in C[a, b]$ by the fractal interpolation function g^* , which is constructed on points of interpolation $\{(x_i, y_i)\}_{i=0}^N$. Thus, it is sufficient to fit parameters $d_i \in (-1, 1)$ to minimize the distance between g and g^* .

We use methods that have been developed for fractal image compression [3]. Notice, that from (3), (2) and (1) follows, that for all $g, h \in L^2[a, b]$

$$\begin{aligned} \|\Phi g - \Phi h\|_2 &= \sqrt{\int_a^b (\Phi g - \Phi h)^2 dx} = \sqrt{\sum_{i=1}^N d_i^2 \int_{x_{i-1}}^{x_i} (g \circ u_i^{-1}(x) - h \circ u_i^{-1}(x))^2 dx} \\ &\leq \max_{i=\overline{1, N}} \{|d_i|\} \cdot \sqrt{\sum_{i=1}^N a_i \int_a^b (g - h)^2 dx} = \max_{i=\overline{1, N}} \{|d_i|\} \cdot \|g - h\|_2. \end{aligned}$$

Thus, $\Phi : L^2[a, b] \rightarrow L^2[a, b]$ is contractive operator with a fixed point g^* .

Furthermore, instead of minimization of $\|g - g^*\|_2$ we will minimize $\|g - \Phi g\|_2$, that makes the problem of optimization much easier. The collage theorem provides validity of such approach.

Theorem 1 *Let (X, d) be a non-empty complete metric space. Let $T : X \rightarrow X$ be a contraction mapping on X with contractivity factor $c < 1$. Then for all $x \in X$*

$$d(x, x^*) \leq \frac{d(x, T(x))}{1 - c}$$

where x^* is the fixed point of T .

► For all integer n we have

$$\begin{aligned} d(x, x^*) &\leq d(x, T(x)) + d(T(x), T^2(x)) + \dots + d(T^{n-1}(x), T^n(x)) + d(T^n(x), x^*) \\ &\leq d(x, T(x))(1 + c + c^2 + \dots + c^{n-1}) + d(T^n(x), x^*). \end{aligned}$$

Letting $n \rightarrow \infty$ we establish the formula. \blacktriangleleft

Considering (1) and (2), we rewrite (3):

$$(\Phi g)(x) = \sum_{i=1}^N \left(\alpha_i(x) - d_i(\beta_i(x) - g \circ \gamma_i(x)) \right) \chi_{[x_{i-1}, x_i]}(x), \quad (4)$$

where

$$\begin{aligned} \alpha_i(x) &= \frac{(y_i - y_{i-1})x + (x_i y_{i-1} - x_{i-1} y_i)}{x_i - x_{i-1}}, \\ \beta_i(x) &= \frac{(y_N - y_0)x + (x_i y_0 - x_{i-1} y_N)}{x_i - x_{i-1}}, \\ \gamma_i(x) &= \frac{(b - a)x + (x_i a - x_{i-1} b)}{x_i - x_{i-1}}. \end{aligned} \quad (5)$$

Thus, we have to minimize functional

$$(\|g - \Phi g\|_2)^2 = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left(g(x) - \alpha_i(x) + d_i(\beta_i(x) - g \circ \gamma_i(x)) \right)^2 dx.$$

Setting partial derivatives with respect to d_i to zero we obtain

$$d_i = \frac{\int_{x_{i-1}}^{x_i} (\alpha_i(x) - g(x))(\beta_i(x) - g \circ \gamma_i(x)) dx}{\int_{x_{i-1}}^{x_i} (\beta_i(x) - g \circ \gamma_i(x))^2 dx}, \quad i = 1, \dots, N. \quad (6)$$

4 Discretization and results

In this section we will approximate discrete data $Z = \{(z_m, w_m)\}_{m=1}^M$, $a = z_1 < z_2 < \dots < z_M = b$ by the fractal interpolation function g^* , which is constructed on points of interpolation $X = \{(x_i, y_i)\}_{i=0}^N$, $N \ll M$. Taking $X \subset Z$, $(x_0, y_0) = (z_1, w_1)$ and $(x_N, y_N) = (z_M, w_M)$ we fit parameters $d_i \in (-1, 1)$ to minimize

$$\sum_{m=1}^M (w_m - g^*(z_m))^2.$$

Let us approximate Z by the piecewise constant function $g : [a, b] \rightarrow \mathbb{R}$. More precisely $g(z) = w_m$, where $(z_m, w_m) \in Z$ and z_m is a nearest neighbor of z . From (6) we obtain the discrete formulas for d_i :

$$d_i = \frac{\sum_{z_m \in [x_i, x_{i+1}]} (\alpha_i(z_m) - w_m)(\beta_i(z_m) - g \circ \gamma_i(z_m))}{\sum_{z_m \in [x_i, x_{i+1}]} (\beta_i(z_m) - g \circ \gamma_i(z_m))^2}, \quad i = 1, \dots, N-1. \quad (7)$$

After finding d_i we obtain formulas for affine transformations A_i and we are able to construct fractal interpolation function g^* for g .

Our aim is to compare fractal approximation with a piecewise quadratic approximation function which is based on the same discretization. On each segment $[x_{i-1}, x_i]$ approximating function has the quadratic form $q_i(x) = k_i x^2 + r_i x + l_i$. To get a continuous function we claim that $q_i(x_{i-1}) = g(x_{i-1})$ and $q_i(x_i) = g(x_i)$. From this we find coefficients k_i and l_i . To find free parameter r_i we minimize functional

$$\sum_{z_m \in [x_{i-1}, x_i]} (w_m - q_i(z_m))^2$$

with respect to r_i on each segment $[x_{i-1}, x_i], i = \overline{1, N}$. The approximating function $q(x)$ will have following form:

$$q(x) = \begin{cases} q_1(x) = k_1 x^2 + r_1 x + l_1, & x \in [a = x_0, x_1]; \\ q_2(x) = k_2 x^2 + r_2 x + l_2, & x \in [x_1, x_2]; \\ \vdots & \\ q_N(x) = k_N x^2 + r_N x + l_N, & x \in [x_{N-1}, x_N = b]. \end{cases}$$

Since there is one free parameter r_i in each function $q_i(x)$ and one parameter d_i for each affine transformation A_i it makes the comparison correct.

To compare fractal and quadratic approximations we consider four types of data.

1. Polynomial function.
2. DNA sequence.
3. Price graph.
4. Random walking graph.

For all types of data $M = 10000$, $z_m = m$, $[a, b] = [1, M]$, $\{w_m\}_{m=1}^M$ are normalized sequences, that is $E(\{w_m\}) = 0$ and $E(\{w_m^2\}) = 1$. For all cases we choose $(x_0, y_0) = (1, w_1)$, $(x_N, y_N) = (M, w_M)$ and other interpolation points (x_i, y_i) , $i = \overline{1, N-1}$ are local extremums of the given data.

Example 2 Let $f(x) = -6x + 5x^2 + 5x^3 - 5x^4 + x^5$, $x \in [-1, 2.5]$. As we work with the segment $[1, M]$ we map $[-1, 2.5]$ to it. Consider sequence $v_m = f\left(\frac{7(m-1)}{2(M-1)} - 1\right)$, $m = \overline{1, M}$. Set $w_m = (v_m - s_1)/s_2$, where s_1 and s_2 are mean and deviation of $\{v_m\}_{m=1}^M$. Figure 2 shows the normalized sequence $\{w_m\}$. Choose five interpolation points $x_0 = 1$, $x_1 = 500$, $x_2 = 4000$, $x_3 = 7500$, $x_4 = 10000$. Applying (7) we obtain $d_1 = 0.066$, $d_2 = 0.155$, $d_3 = 0.033$, $d_4 = 0.096$. The small values of $|d_i|$ mean that on segments $[x_{i-1}, x_i]$ fractal approximation function looks as a straight line. Figure 3 shows the graphs of fractal and quadratic approximating functions.

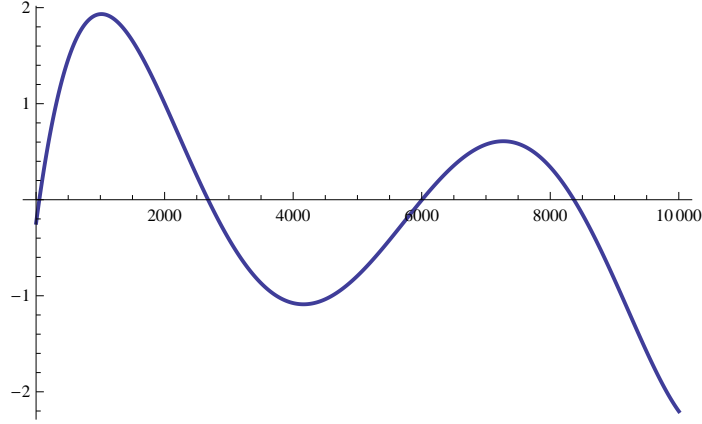


Figure 2: The graph of original function g .

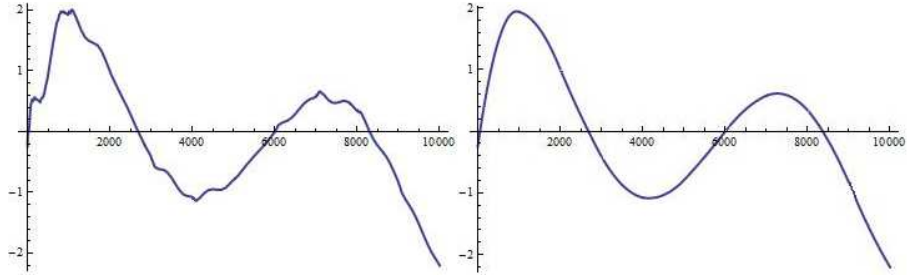


Figure 3: Fractal and quadratic interpolations of the polynomial function.

Example 3 A DNA sequence can be identified with a word over an alphabet $\mathcal{N} = \{A, C, G, T\}$. Here we have the sequence of 10000 nucleotides of *Edward-siella tarda*. The graph represented by the formula

$$v_1 = 0, v_m = v_{m-1} + \begin{cases} +1, & \text{if } m^{\text{th}} \text{ nucleotide belongs to } (A, G); \\ -1, & \text{if } m^{\text{th}} \text{ nucleotide belongs to } (C, T). \end{cases}$$

For full description of representation of DNA primary sequences see [4]. Figure 4 shows the sequence $\{w_m\}$ after normalization of $\{v_m\}$ according to the formula in the previous example. Interpolation points are $x_0 = 1, x_1 = 1000, x_2 = 2500, x_3 = 3000, x_4 = 3500, x_5 = 5000, x_6 = 6500, x_7 = 7000, x_8 = 8000, x_9 = 9000, x_{10} = 10000$. Applying (7) we obtain $d_1 = -0.001, d_2 = 0.274, d_3 = 0.31, d_4 = 0.24, d_5 = -0.057, d_6 = 0.211, d_7 = -0.42, d_8 = -0.121, d_9 = 0.215, d_{10} = 0.158$. Figure 5 shows the graphs of fractal and quadratic approximating functions.

Example 4 We take price wave of 10000 prices $v_m, m = \overline{1, M}$ of one day period for EUR/USD, then normalize it (Figure 6). Interpolation points are

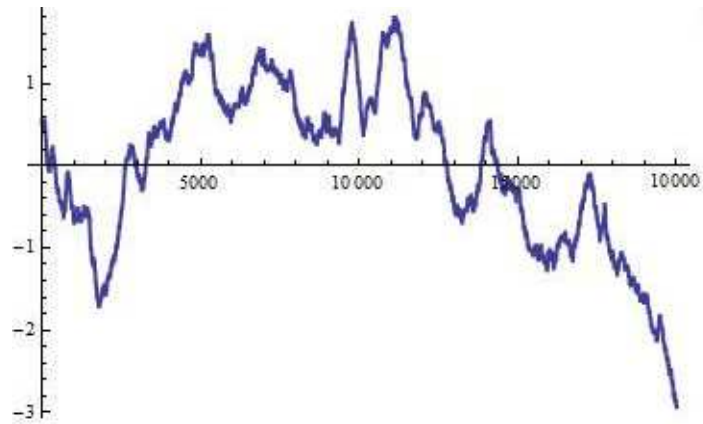


Figure 4: Picture shows DNA Graph of 10000 nucleotides of *Edwardsiella tarda*.

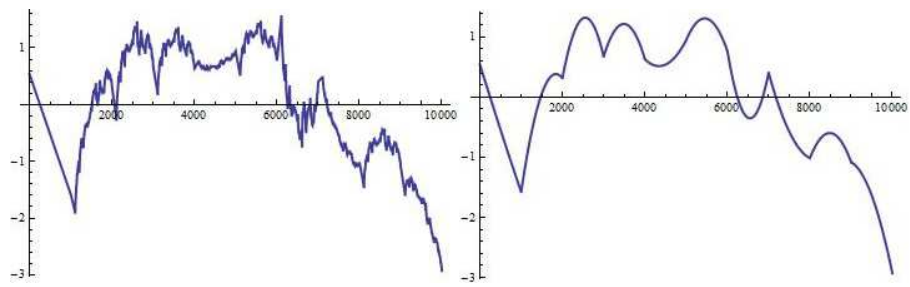


Figure 5: Fractal and quadratic interpolations of the DNA Graph.

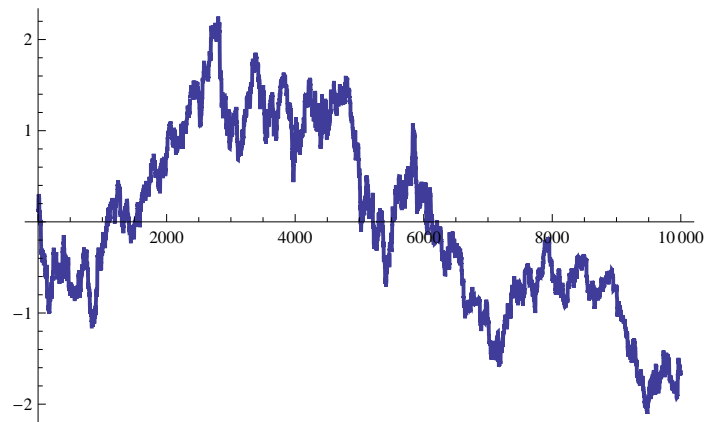


Figure 6: Picture shows Price Graph for EUR/USD.

$x_0 = 0, x_1 = 500, x_2 = 1500, x_3 = 2000, x_4 = 2500, x_5 = 3000, x_6 = 4000, x_7 = 5000, x_8 = 6000, x_9 = 8000, x_{10} = 10000$. Applying (7) we obtain $d_1 = -0.334, d_2 = -0.004, d_3 = 0.315, d_4 = 0.307, d_5 = 0.333, d_6 = -0.28, d_7 = -0.067, d_8 = 0.027, d_9 = 0.047, d_{10} = -0.33$. Figure 7 shows the graphs of fractal and quadratic approximating functions.

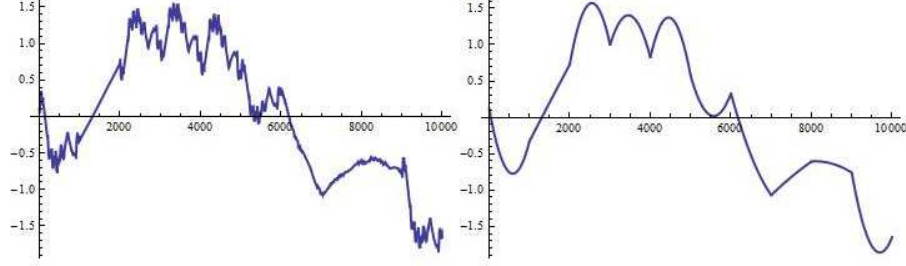


Figure 7: Fractal and quadratic interpolations of the Price Graph.

Example 5 Picture shows Random Walking Graph. It represented by the formula $v_0 = 0, v_i = v_{i-1} + \xi_i$, where ξ_i is a random value with normal distribution. Interpolation points are $x_0 = 0, x_1 = 1500, x_2 = 2000, x_3 = 3000, x_4 = 4000,$

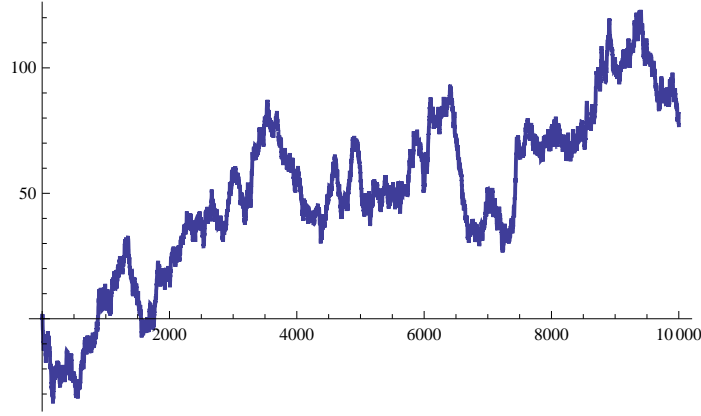


Figure 8: Normalized Random Walking graph.

$x_5 = 5500, x_6 = 6300, x_7 = 7600, x_8 = 8000, x_9 = 9000, x_{10} = 10000$. Applying (7) we obtain $d_1 = -0.237, d_2 = 0.14, d_3 = -0.020, d_4 = -0.105, d_5 = 0.105, d_6 = 0.0545, d_7 = -0.184, d_8 = -0.368, d_9 = 0.081, d_{10} = -0.111$. Figure 9 shows the graphs of fractal and quadratic approximating functions.

To compare the results we calculate approximation errors for each type of data. Let $h(x)$ be the approximating function for data $\{w_m\}_{m=1}^M$. Then ap-

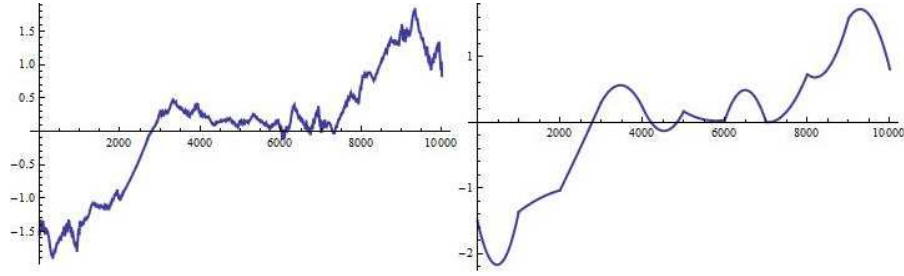


Figure 9: Fractal and quadratic interpolations of the Random Walking graph.

proximation error is

$$\sqrt{\sum_{m=1}^M \frac{(h(x_m) - w_m))^2}{M}}.$$

Here we represent the table of approximation errors for each type

	<i>Fractal</i>	<i>Quadratic</i>
<i>PolynomialFunction</i>	0.0359037	0.0245094
<i>DNA Primary Sequence</i>	0.0692072	0.0624714
<i>Price Graph</i>	0.0501345	0.0533686
<i>Random Walking</i>	0.1015339	0.101438

From it we see, that fractal approximation is better for price graph and nearly equal for random walking, but much worse for smooth function and slightly for DNA sequence. Different results were appearing during calculations of errors. We assume that some conditions could give us more exact approximation results from fractal interpolation function and for that extra observations should be established.

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